Mid-term Test
SOLID MECHANICS (NASM)
October 1, 2024, 11:00-11:45 h

NB This is a *closed-book* exam. It comprises four problems, for which one can obtain the following points:

Question	# points
1	2
2	3
3	2
4	1+1=2

The number of points is indicated next to each subquestion inside a rectangular box in the right-hand margin on the next pages.

The mid-term grade is calculated as $9 * (\# \text{ points})/9 + 1$ and contributes 30% to the final grade.

Question 1 Prove that the outcome of the quadratic product $\boldsymbol{x} \cdot \boldsymbol{A} \cdot \boldsymbol{x}$ for an arbitrary vector $\boldsymbol{x}$ is independent of the skew-symmetric part of the tensor $\boldsymbol{A}$. [2]

Question 2 In two dimensions, let the components of the deformation gradient tensor be specified by the matrix

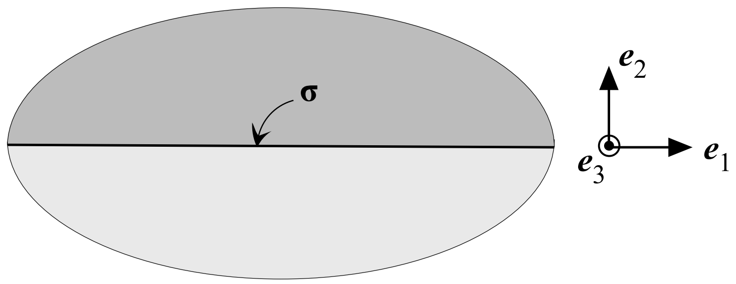
$$[F_{ij}] = \begin{bmatrix} 1 & 0 \\ \gamma & 1 \end{bmatrix}.$$

a. Show that the corresponding components of the strain tensor in the geometrically linear theory are given by [1]

$$[\varepsilon_{ij}] = \begin{bmatrix} 0 & \gamma/2 \\ \gamma/2 & 0 \end{bmatrix}.$$

b. Is the inverse also correct? In other words: given these strain components, does this imply that the deformation gradient is as specified above? Explain your answer. [2]

Question 3 Consider a flat interface between two dissimilar materials in a loaded composite material. When the system is in equilibrium, which components of the stress tensor $\boldsymbol{\sigma}$ in three dimensions are continuous across this interface (with unit normal $\boldsymbol{n} = \boldsymbol{e}_2$)? [2]



Question 4 Consider a planar strain state characterized by principal strains ε_1 and ε_2 with respect to the basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$; that is,

$$[\varepsilon_{ij}] = \begin{bmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{bmatrix}.$$

When viewed from the vantage point of a basis $\{\boldsymbol{e}_a^*\}$ $(a = 1,2)$ that is rotated over an angle θ in the plane, the strain components ε_{ab}^* will no longer be principal strains and shear strain components will emerge.

a. At what value of θ will the shear strain component ε_{12}^* be maximal? [1]

b. What is the normal strain component at this particular orientation? [1]

Solutions Mid-term Test
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Question 1
Alternatively:

a. Recognize that $\boldsymbol{x}\cdot\boldsymbol{A}\cdot\boldsymbol{x} = \boldsymbol{A} : (\boldsymbol{x}\otimes\boldsymbol{x})$, with $\boldsymbol{x}\otimes\boldsymbol{x}$ a symmetric second-order tensor; followed by recalling that the scalar product of a symmetric and a skewsymmetric tensor vanishes. [1] [1]

b. Introduce symmetric and skew-symmetric parts $\boldsymbol{S}$ and $\boldsymbol{W}$, respectively, and expand:

$$\boldsymbol{x}\cdot\boldsymbol{A}\cdot\boldsymbol{x} = \boldsymbol{x}\cdot\boldsymbol{S}\cdot\boldsymbol{x} + \boldsymbol{x}\cdot\boldsymbol{W}\cdot\boldsymbol{x};$$

exploit the skewsymmetry of $\boldsymbol{W}$ to derive [1]

$$\boldsymbol{x}\cdot\boldsymbol{W}\cdot\boldsymbol{x} = x_i W_{ij} x_j = -x_i W_{ji} x_j = -x_j W_{ij} x_i = -x_i W_{ij} x_j = -\boldsymbol{x}\cdot\boldsymbol{W}\cdot\boldsymbol{x} = 0\,.$$

[1]

Question 2

a. Since $F_{ij} = \delta_{ij} + u_{i,j}$, the only-nonzero component of the displacement gradient is $u_{2,1} = \gamma$. Substitution into $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ yields the specified strain components. [1]

b. No, the inverse is not correct, since there are many displacement gradients that satisfy $\frac{1}{2}(u_{1,2} + u_{2,1}) = \gamma/2$. More specifically, there are many rotations $\omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i})$ that can be added to the given state of strain ε_{ij} in order to arrive at the displacement gradient. The one given in the question corresponds to a particular rotation: $\omega_{12} = -\gamma/2$. [2]

Question 3
It is the traction across the interface that needs to be continuous, as a consequence of Newton's third law (action = −reaction). Since [1]

$$\boldsymbol{t} = \boldsymbol{\sigma}\cdot\boldsymbol{n} = \sigma_{i2}\boldsymbol{e}_i\,,$$

the components $\{\sigma_{12}, \sigma_{22}, \sigma_{23}\}$ are continuous (note that the normal stresses parallel to the interface, i.e. σ_{11} and σ_{33} can change discontinuously) [1]

Question 4
This question is most easily answered with reference to the elementary Mohr's circle, but also through the transformation formulas.

a. When $2\theta = 90^\circ$, hence $\theta = 45^\circ$. [1]

b. $\frac{1}{2}(\varepsilon_1 + \varepsilon_2)$ [1]